Temperature Oscillations and Vibrations of AC Resistively Heated Thin Filaments

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Resistively heated filaments are used to deposit diamond or to act as a substrate in the manufacture of ceramic fibers. This article analyzes the effects of an AC power source, in particular the oscillation in filament temperature and its influence on the kinetic rates, as well as the vibration of the filament, when it is mounted in a state of tensile stress. These effects diminish when the filament gauge is large enough and the analysis can help to decide whether an AC or a DC power source should be used.

Introduction

Two important technologies that use heated filaments are the deposition of diamond films and the manufacture of ceramic fibers. A simple, but popular, method to deposit diamond films is described by Jansen et al. (1990). A filament is mounted between two electrodes (either coiled or in tension) and heated to 2,000–2,300°C. A gaseous mixture, containing hydrocarbons, flows over the filament onto a separate substrate that is heated between 500–900°C. The filament produces electrons by thermionic emission. The electrons interact with the hydrocarbons to form intermediate species that adsorb on the substrate. The temperature of the filament will oscillate and the amplitude is determined by the filament gauge. Since thermionic emission is described by the Richardson equation, which has an exponential dependency on temperature, one can expect a fluctuation in the flux of emitted electrons.

For the manufacture of ceramic fibers, a thin filament is mounted between two electrodes and heated to 1,150–1,350 K. A schematic of the experimental configuration is shown in Figure 1. A gaseous mixture containing the reactive precursors flows axially along the fiber and reacts on the fiber surface to form a protective ceramic coating (see Scholtz et al., 1991, for more details.) Experiments carried out in the Laboratory for Ceramic and Reaction Engineering showed that the filament underwent oscillations when an AC source was used. The oscillations changed in amplitude and wavenumber during the course of the reaction. These observations prompted our investigation to get a better understanding of the effects of an AC-source on the system.

Temperature oscillations on the filament surface will lead to oscillations in the deposition rate, also due to the temperature dependency of the Arrhenius kinetics. If the filament is mounted in a tensile state, it can also begin to vibrate in any of its natural harmonic modes with disastrous effects on the quality of the deposition. Therefore, investigators will only use DC power sources for this application. But, it is still important to analyze the problem of AC heating, because it is more economical to use, especially for larger production and for the heating of bigger workpieces. For filaments thicker than a certain gauge, these oscillatory effects become negligible and AC heating can be used.

The vibrations in the filament are induced by the temperature oscillations. An interesting nonlinear effect now becomes apparent. The rate of deposition varies along the axial position, changing the mass and mechanical properties locally. If it is recognized that the material properties of the deposit differ from the substrate, then it becomes evident that the problem can be very complex. The thermoelastic equations also become coupled when the vibrations are large. However, this complexity is not addressed in this article. The thermoelastic model

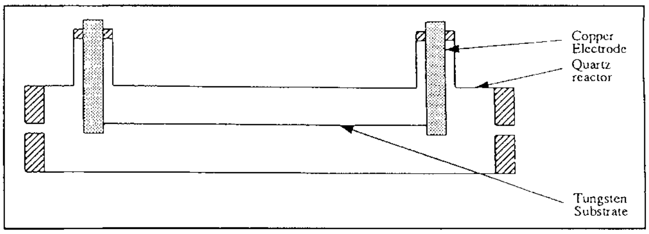

Figure 1. Reactor for ceramic fiber manufacture.

is discussed, and typical system parameters are listed. In addition, the effects of temperature oscillations on thermionic emission and rate of chemical reactions are presented, as well as the problem of temperature-induced vibrations and the application of this work.

Model

We want to study the temperature oscillations and vibrations, prior to any deposition on the filament's surface. The energy conservation equation is:

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} - 2\frac{U}{R}(T - T_o) + \frac{I^2 \rho_R}{A^2}\sin^2(\omega t) \quad (1)$$

The aspect ratio of the filament can vary anywhere between 1,000 and 30,000, and the lumping of the conduction in the radial direction is justified. The independent variable, x, denotes the axial distance between the two electrodes. In cases where the amplitude of the vibrations become large, x should be replaced by the arc length of the filament. U is the overall heat transfer coefficient to the surrounding medium and it includes both convection and radiation:

$$U = h_{\text{conv}} + 4\sigma_{SB} T^3 \Omega$$

where Ω is a view factor for the specific geometry. The Joule effect term contains the resistivity ρ_R, ω is 50 Hz, and A is the cross-sectional area of the filament.

The displacements of the filament are described by the theory of elastica. Thermal moments of these filaments are negligible and, since we assume that the filament will be mounted in a state of tension, allow us to use the string equation. Let (y, x) denote the transversal and axial position of a filament element, then the equation can be written as:

$$\frac{P}{\kappa^2}\frac{\partial^2 y}{\partial x^2} + \frac{1}{\kappa^2}\frac{\partial P}{\partial x}\frac{\partial y}{\partial x} - \frac{P\left(1 + \frac{\partial y}{\partial x}\right)}{\kappa^4}\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2} + \text{Damping} \quad (2)$$

where P is the stress in the filament (net tension) and κ is defined as:

$$\kappa = \left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{\frac{1}{2}}.$$

The damping term includes the product of two first derivatives with respect to time and a drag coefficient, but it will not be considered in our formulation.

The constitutive equation that will be used in this study is the linear thermoelastic formulation:

$$P = P_s - E_y \alpha (T - T_{\text{mean}}) \quad (3)$$

where E_y is Young's modulus, α is the linear thermal expansion coefficient. T_{mean} is the time-averaged profile around which the temperature oscillates and P_s is the applied tension at the median temperature.

Let L denote the distance between the two electrodes and P_o the maximum tensile strength, the following nondimensional variables can be introduced:

$$\sigma = \frac{P}{P_o}, \ \theta = \frac{T - T_o}{T_o}, \ \tau = \omega t, \ (\xi, \eta) = \frac{(x, y)}{L}.$$

Equations 1-3 can be written in nondimensional form as:

$$\frac{\partial \theta}{\partial \tau} = \epsilon \frac{\partial^2 \theta}{\partial \xi^2} - a\theta + \frac{b}{2}[1 - \cos(2\tau)] \quad (4)$$

$$\frac{\sigma}{\kappa^2}\frac{\partial^2 \eta}{\partial \xi^2} + \frac{1}{\kappa^2}\frac{\partial \sigma}{\partial \xi}\frac{\partial \eta}{\partial \xi} - \frac{\sigma}{\kappa^4}\left(1 + \frac{\partial \eta}{\partial \xi}\right)\frac{\partial \eta}{\partial \xi}\frac{\partial^2 \eta}{\partial \xi^2} = \lambda\frac{\partial^2 \eta}{\partial \tau^2} \quad (5)$$

$$\sigma = \sigma_s - \gamma(\theta - \theta_{\text{mean}}) \quad (6)$$

If $|\partial\eta/\partial\xi| \ll 1 (\kappa \approx 1)$, the form of the energy equation is correct. The time scale used in Eq. 4 is 0.02 s. Because the filament has a very small heat capacity, it responds to the changes in the heating rate. As the filament's heat capacity decreases, this response becomes more pronounced. The larger the oscillations of the temperature profile becomes, the more will the resistivity and other physical properties of the filament change during a cycle. Constant physical properties, however, are used in the model and they are evaluated at the median temperature at the center of the filament.

There are five parameters in the model, which are instructive to evaluate for a typical filament material.

Table 1. Physical Properties

k	$\rho \times 10^3$	C_p	$\alpha \times 10^6$	$\rho_R \times 10^8$	$E_y \times 10^{-6}$	$P_o \times 10^{-3}$
116.33	19.263	338.4	5.5	30.81	364	700

Values of Parameters

Tungsten is an important filament material, because it has high melting point and is used in both diamond deposition and ceramic fiber manufacture. In Table 1, its properties are listed at a temperature of 1,200 K. Filament diameters of 10, 50 and 100 μm are considered. In Table 2, the parameter values are reported for these three different sizes. The value of U was taken as 250 $W/(m^2 \cdot K)$ (see Scholtz et al., 1991) and the mounting distance was taken as $L = 0.3$ m.

Temperature Oscillations

As was pointed out earlier, we assume that $\kappa \approx 1$ and the energy balance can be written as in Eq. 4. Also the contribution

Table 2. Parameter Values

Dia.	a	b	ϵ	λ	γ
10 μm	3.07 (−1)	1.842	3.97 (−6)	6.2 (−3)	8.58 (−1)
50 μm	6.14 (−2)	3.684 (−1)	3.97 (−6)	6.2 (−3)	8.58 (−1)
100 μm	3.07 (−2)	1.842 (−1)	3.97 (−6)	6.2 (−3)	8.58 (−1)

of strain heating is neglected in the energy balance, since its contribution will be small compared to the other terms. Then, the thermoelastic equations become uncoupled and the temperature can be solved independently.

The boundary conditions for temperature are taken as:

$$\theta = 0 \text{ at } \xi = 0,1 \tag{7}$$

The ends are clamped in electrodes that are larger than the filament and act as heat sinks, maintaining the temperature at ambient values at the end. A particular solution of Eq. 4 is:

$$\theta_p = \frac{b}{2a}\left[1 - e^{-a\tau} - \frac{\cos(2\tau) - e^{-a\tau} + \frac{2}{a}\sin(2\tau)}{1 + \frac{4}{a^2}}\right] \tag{8}$$

The asymptotic behavior is given by:

$$\theta_{p\infty} = \frac{b}{2a}\left[1 - \frac{\cos(2\tau) + \frac{2}{a}\sin(2\tau)}{1 + \frac{4}{a^2}}\right] \tag{9}$$

The remainder of the asymptotic solution is considered in two parts. The first one is:

$$\theta_{H1} = -\frac{b}{2a}\,\frac{\sinh[\beta(\xi - 1)] - \sinh(\beta\xi)}{\sinh(\beta)} \tag{10}$$

where $\beta = (a/\epsilon)^{0.5}$.

The second term θ_{H2} satisfies the equation:

$$\frac{\partial\theta}{\partial\tau} = \epsilon\frac{\partial^2\theta}{\partial\xi^2} - a\theta \tag{11}$$

subject to the boundary conditions:

$$\theta_{H2} = \frac{b}{2a}\,\frac{[\cos(2\tau) + \frac{2}{a}\sin(2\tau)]}{1 + \frac{4}{a^2}} \text{ at } \xi = 0,1. \tag{12}$$

θ_{H2} is written as the sum of two complex functions:

$$\psi_1 = f_1 e^{2i\tau}$$

$$\psi_2 = f_2 e^{i(2\tau - \pi/2)}$$

Substituting these forms into Eq. 11 and only considering the real parts, one finally obtains the following expression for θ:

$$\theta = \frac{b}{2a}\left[1 - \frac{\cos(2\tau) + \frac{2}{a}\sin(2\tau)}{1 + \frac{4}{a^2}}\right] + \frac{b}{2a}\,\frac{\sinh[\beta(\xi - 1)] - \sinh(\beta\xi)}{\sinh(\beta)}$$

$$+ \frac{b}{2a\left(1 + \frac{4}{a^2}\right)}\left\{e^{v\xi - v}[\cos(w)\cos(w\xi) + \sin(w)\sin(w\xi)]\left[\cos(2\tau) + \frac{2}{a}\cos\left(2\tau - \frac{\pi}{2}\right)\right]\right.$$

$$- e^{v\xi - v}[\cos(w)\sin(w\xi) - \sin(w)\cos(w\xi)]\left[\sin(2\tau) + \frac{2}{a}\sin\left(2\tau - \frac{\pi}{2}\right)\right]$$

$$+ e^{-v\xi}\{\cos(w)\cos[w(1 - \xi)] + \sin(w)\sin[w(1 - \xi)]\}\left[\cos(2\tau) + \frac{2}{a}\cos\left(2\tau - \frac{\pi}{2}\right)\right]$$

$$\left. - e^{-v\xi}\{\cos(w)\sin[w(1 - \xi)] - \sin(w)\cos[w(1 - \xi)]\}\left[\sin(2\tau) + \frac{2}{a}\sin\left(2\tau - \frac{\pi}{2}\right)\right]\right\} \tag{13}$$

where $(v + iw) = [(a/\epsilon) + i(2/\epsilon)]^{0.5}$.

The amplitude will decrease from the center of the filament toward the ends, which are clamped in electrodes. In Figure 2, the oscillations in the temperature at the center of a 100-μm filament are shown. The temperature oscillates with twice the forcing frequency and the amplitude at the center of the filament is 30 K. When the filament diameter is 10 μm, the amplitude increases tenfold to 300 K. This result is shown in Figure 3.

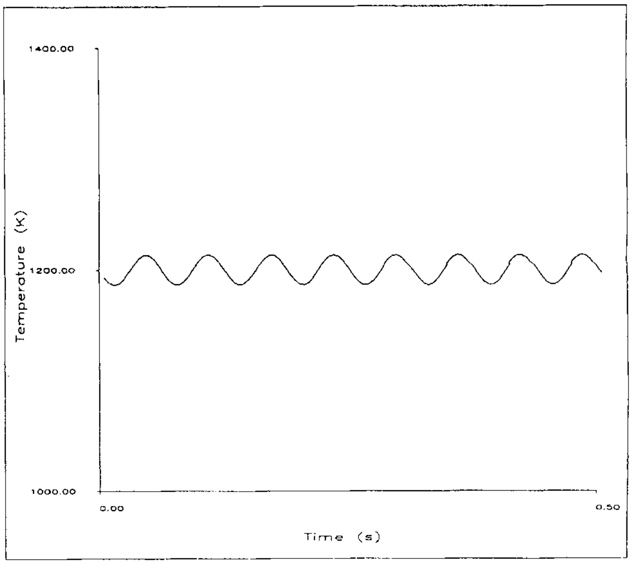

Figure 2. Temperature oscillations of 100-μm·W filament.

These oscillations will influence temperature-dependent processes on the surface of the filament. First, we will consider thermionic emission that is described by the Richardson equation:

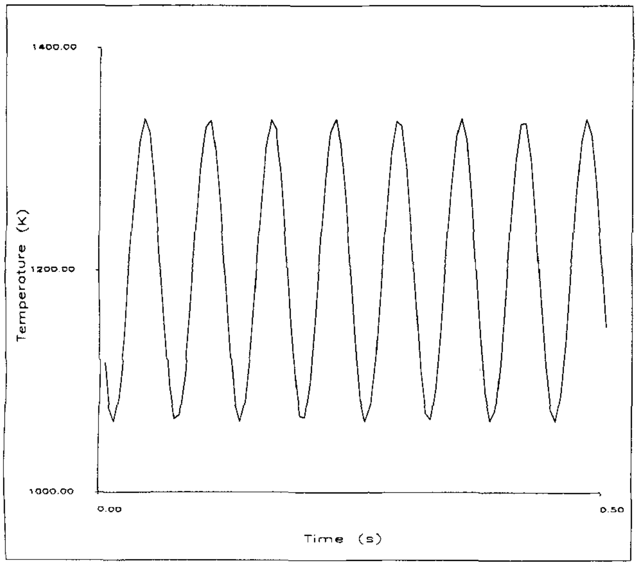

Figure 3. Temperature oscillations of 10-μm·W filament.

$$P=R_c T^2 e^{-\frac{\phi}{k_B T}}. \tag{14}$$

The Richardson constant R_c for tungsten at temperatures between 1,900-2,100 K is 8.9×10^{-3} Amp/m^2/K^2, and the work function ϕ is 4.55 eV (Fomenko, 1966, p. 54). In Figure 4, the variation in flux is shown for a tungsten filament of 10 μm in diameter, heated to an average temperature of 2,100 K. The flux varies periodically as expected, but the maxima, which appears as sharp peaks, exceed the average value by one order of magnitude. When the temperature drops below the average, the flux decreases by more than an order of magnitude, thence demonstrating the temperature sensitivity of these processes.

The second example we will use to demonstrate the effect of oscillating temperatures is the coating of tungsten fibers by

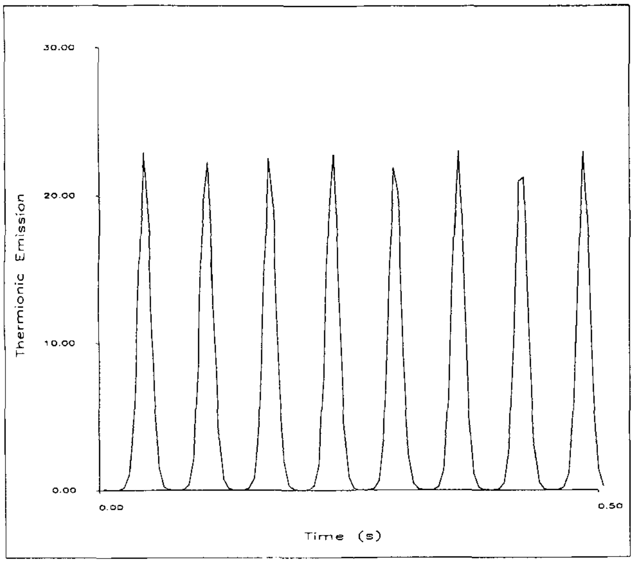

Figure 4. Thermionic emission rate of 10-μm filament.

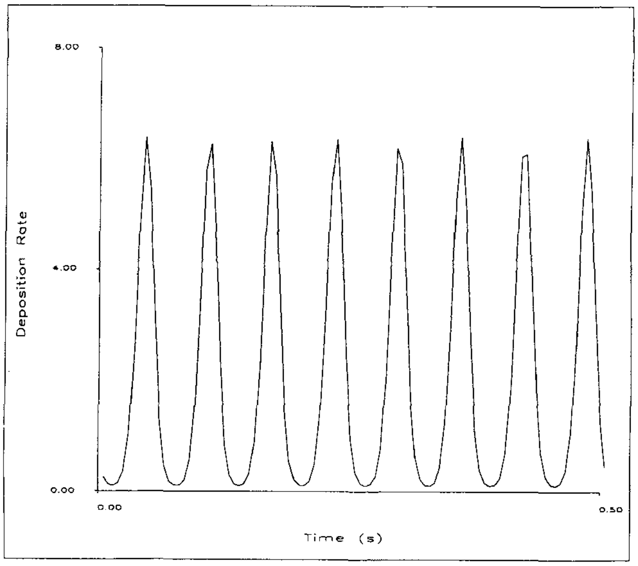

Figure 5. Reaction rate on surface of 10-μm filament.

boron, using BCl_3 and H_2 as precursors. For details of the process and the kinetics in particular, see Scholtz et al. (1991). The rate of boron deposition is described by:

$$R_B=17\cdot e^{-15{,}600/T}(g_1-g_2)\ \text{kmol/m}^2\text{/s/kPa} \tag{15}$$

where g_1 and g_2 denote forward and backward reaction rates:

$$g_1=\frac{K^1_{eq}x_{BCl_3}x_{H_2}}{x_{HCl}(1/p-K_{HCl}x_{HCl})} \tag{16}$$

$$g_2=\frac{x^2_{HCl}}{K^2_{eq}x_{H_2}(1/p+K_{HCl}x_{HCl})}\ . \tag{17}$$

For illustration, we will consider average values for all molar fractions, $x_{BCl_3}=0.05$, $x_{H_2}=0.949$, and $x_{HCl}=0.001$. Then, the following relationship results:

$$R_B=17e^{-15{,}600/T}\left[x_{BCl_3}x_{H_2}\frac{e^{2-5{,}640/T-6.8\times 10^8/T^3}}{x_{HCl}(0.01-0.0249x_{HCl}e^{4{,}982/T})}\right] \tag{18}$$

$$-\frac{x^2_{HCl}}{x_{H_2}e^{[0.37\exp(0.0069T)-3{,}321]/T}(0.01+0.0249x_{HCl}e^{4{,}982/T})} \tag{19}$$

Using the oscillating temperature shown in Figure 3, the relative rate can be calculated, as shown in Figure 5. The relative rate is defined as $[R_B(T)]/[R_B(1{,}200)]$. The rate has a maximum of 6.3 times the value $R_B(1{,}200)$ and decreases to one tenth of this value. Note that the time scale in the figure presents real time. The period of the oscillations in all cases is 1/16 of a second ($2\pi/100$).

These two examples clearly illustrate the important role that an AC power source can play in a process. Whenever the sample heated is small, its thermal response should be carefully considered.

Thermally Induced Vibrations

In the previous section we have discussed the thermal oscillation in the filament. If the filament is mounted such that during all times a net tension is maintained, the filament can potentially start to vibrate. What happens in practice is that the filament is tightened between the electrodes before the current is switched on. After the required temperature is reached, the filament is usually slack and tightened again before deposition is started (in the case of ceramic fiber manufacture). The second tightening puts a tension on the filament, and if any vibrations are observed, the tension is further increased. This arbitrary tightening of the filament can be eliminated, if one properly analyzes the stability of the governing equation. It is also important to know the tensile load during deposition, because it will effect the final strength of the ceramic fiber.

The assumption of $\kappa \approx 1$ will hold best for the 100-μm filament, and the analysis will be done for this case. Therefore, the vertical displacements are small enough to validate the use of the linear form of Eq. 5 and one can write:

$$\sigma \frac{\partial^2 \eta}{\partial \xi^2} = \lambda \frac{\partial^2 \eta}{\partial \tau^2} \tag{20}$$

where σ is given by Eq. 6. Equation 20 has the form of the wave equation. The problem can be somewhat simplified, if one only uses the first righthand side term of Eq. 13 to replace the temperature function in Eq. 20. The dominant role of this term is understood, if one realizes that axial conduction is negligible along most parts of the filament and with the exception of a small region near the electrodes, the other terms are negligible. Considering only the first term of Eq. 13, the string equation can be written:

$$\left\{ \sigma_s + \gamma \frac{b}{2a} \left[\frac{\cos\left(2\tau + \frac{2}{a}\sin(2\tau)\right)}{1 + \frac{4}{a^2}} \right] \right\} \frac{\partial^2 \eta}{\partial \xi^2} = \lambda \frac{\partial^2 \eta}{\partial \tau^2} \tag{21}$$

The natural harmonics of the filament are $\sin(\pi n \xi)$, $n = 1,2...$ and one can write Eq. 21 as:

$$\frac{d^2\eta}{d\nu^2} = -\left\{ \sigma_s + \frac{2\Delta}{1 + \frac{4}{a^2}} \left[\cos\left(\frac{2\lambda^{0.5}\nu}{\pi n}\right) + \frac{2}{a}\sin\left(\frac{2\lambda^{0.5}\nu}{\pi n}\right) \right] \right\} \eta \tag{22}$$

where $\tau = \frac{\lambda^{0.5}}{\pi n}\nu$ and $\Delta = \frac{\gamma b}{4a}$.

This equation is easily recognized as a form of the Mathieu equation, the stability of which has been studied by several authors (cf. Bender and Orszag, 1978). The solution of Eq. 22 can be found in the form of a perturbation series. Using a as an expansion parameter (cf. Table 2), η is sought in the form:

$$\eta = \sum a^n \eta_n \tag{23}$$

The critical values of σ_s, where η grows unstable, can be determined from the Fredholm solvability condition. First, note that σ_s is positively bounded from below:

$$\sigma_s > K = \frac{\gamma b \left\{ \cos\left[\tan^{-1}\left(\frac{2}{a}\right)\right] + \frac{2}{a}\sin\left[\tan^{-1}\left(\frac{2}{a}\right)\right] \right\}}{2a\left(1 + \frac{4}{a^2}\right)}$$

and from above by the maximum tensile strength: $\sigma_s \in (K,1) = S$.

The linear operator $L = (d^2/d\nu^2) + \sigma_s$ is self-adjoint, and the Fredholm solvability criterion can be easily applied to the expansion. The oscillatory solution will be stable except at:

$$\sigma_{cj} = \frac{\lambda j^2}{\pi^2 n^2} \tag{24}$$

For a fixed n, one would be interested only in those $\sigma_{cj} \in S$. The critical values of σ_s have been identified, but the stability boundaries in a-σ_s plane must be determined. σ_{cj}'s can be expanded as powers of a, and by defining multiple scales of time (ν and $\mu = a\nu$), one obtains the following result, corrected to first order in a:

$$\sigma_{cj} = \frac{\lambda j^2}{\pi^2 n^2} \pm a \tag{25}$$

For the 100-μm filament, the lower bound on σ_s is $K = 0.0395$. Since $\sigma_{cj} \in S$, it follows from Eq. 24 that $j \geq 8$ (associated with first harmonic). Now let us consider this minimum case, $j = 8$. For $\sigma_s = 0.04$, the system falls within the unstable region:

$$\frac{\lambda 8^2}{\pi^2} - a \leq \sigma_s \leq \frac{\lambda 8^2}{\pi^2} + a$$

and Eq. 22 was integrated by Gear's method. The temporal behavior of η vs. time is shown in Figure 6. The system clearly

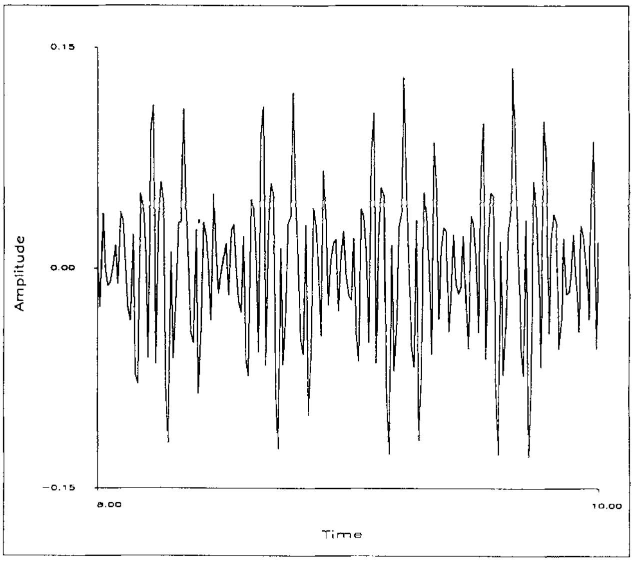

Figure 6. Amplitude vs. time for 28-MPa tensile load.

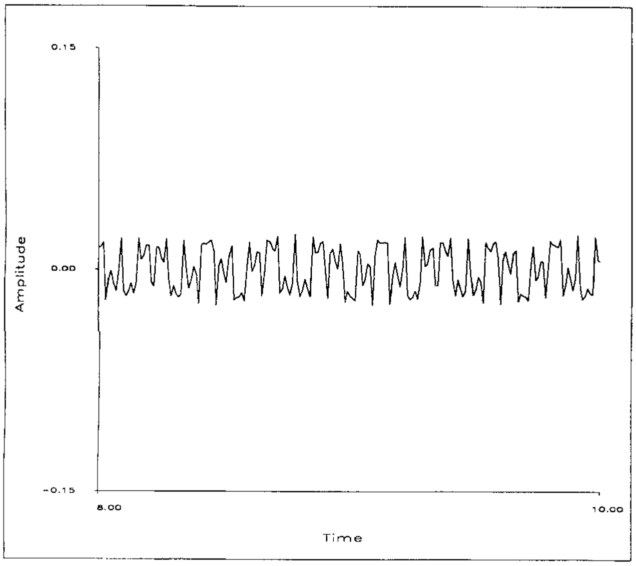

Figure 7. Amplitude vs. time for 56-MPa tensile load.

is unstable. The nonlinear terms, which are not included in Eq. 22, will saturate the instability, but this is exemplary of the cases observed in the laboratory when violent oscillations occurred. In Figure 7, the behavior is shown for an increased tension $\sigma_s = 0.08$, which lies on the stable side of the threshold given by Eq. 25. The amplitude settles in a pattern with a constant maximum peak. Of course, the real system is dissipative and the amplitude will slowly decrease due to damping. But, the filament has a 56-MPa tensile load on it ($P_o \times 0.08$) and this will compromise the total strength of the filament.

Conclusions

An analysis is presented for heating a filament by an AC power source. An exact expression is derived for the spatial distribution of temperature at any point in time. Examples are presented of physical/chemical processes that are modulated by the oscillating temperature, and it can be concluded that the processes can be considerably modified, especially for thin filaments. The maximum amplitude of the temperature oscillation is:

$$\Delta T_{\max} = \frac{16 I^2 \rho_R}{\pi^2 D^3 U} \frac{\left\{ \cos\left[\tan^{-1}\left(\frac{2}{a}\right)\right] + \frac{2}{a} \sin\left[\tan^{-1}\left(\frac{2}{a}\right)\right]\right\}}{\left(1 + \frac{4}{a^2}\right)} \tag{26}$$

Equation 26 should prove useful to people working with resistively heated filaments, and it can help to determine the deviations of temperature-related processes that occur on the filament surface.

An important simplification that was made in this analysis was the use of a constant resistivity, evaluated at the mean temperature. If it is recognized that resistivity usually has the functional form, $\rho_R = \rho_R(T)$, the system could have more complex dynamic behavior, which can lead to a very interesting study. The resistivity of graphite fibers, which are also used for ceramic fiber manufacture, is a highly nonlinear function of temperature. In general, when temperature-dependent properties are considered, Eq. 4 is no longer linear and the external forcing can lead to more complex behavior. Another potentially interesting problem is the nonlinear coupling between the parabolic temperature equation and the hyperbolic displacement equation when large displacements are present.

An approximate form is derived to describe the vibration of the filament, due to thermal action. Critical values of prestressing are determined to eliminate vibrations. The oscillating solution is stable for $\sigma_s \neq \sigma_{cj}$ and for a sufficiently small value of a in the stable regions of the $a - \sigma_s$ plane.

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Notation

a = $2U/(R\omega\rho C_p)$
A = filament cross-section, m^2
b = $(I^2\rho_R)/(T_o A^2 \omega\rho C_p)$
C_p = specific heat, J/kg/K
D = diameter of filament, $2R$
E_y = Young's modulus, kPa
$g_{1,2}$ = rate of reactions, defined in Eqs. 16 and 17
I = current, C/s
k = thermal conductivity, W/m/K
k_B = Boltzmann constant, eV/K
L = distance between electrodes, m
n = wave number of vibrating string
P = stress (tensile), kPa
P_o = tensile strength of material, kPa
P_s = applied tensile stress at mean temperature, kPa
R = radius of filament, m
R_c = Richardson constant, C/s/m^2/K^2
t = time
T = temperature, K
U = heat transfer coefficient, W/m^2/K
x = space variable, m
y = vertical displacement, m

Greek letters

α = linear expansion coefficient, 1/K
β = $(a/\epsilon)^{0.5}$
γ = $(E_y \alpha T_o)/P_o$
Δ = $\gamma b/4a$
ϵ = $k/(\rho C_p \omega L^2)$
η = dimensionless vertical displacement, y/L
θ = dimensionless temperature $(T - T_o)/T_o$
κ = defined as $\sqrt{[1 + (\partial y/\partial x)^2]}$
λ = $(\rho L^2 \omega^2)/P_o$
μ = slow time scale $a\nu$
ν = $(\tau\pi n)/\lambda^{0.5}$
ξ = dimensionless space variable, x/L
ρ = density of filament material, kg/m^3
ρ_R = resistivity of filament material, ohm·m
σ = dimensionless stress P/P_o
σ_s = dimensionless applied stress P_s/P_o
σ_{SB} = Stefan-Boltzmann constant
τ = $(tU^2\rho C_p)/k$
ϕ = work function, eV
ψ = complex functions
ω = frequency of AC source, Hz

Subscripts

cj = jth critical value
$H1$, $H2$ = first and second terms of homogeneous solution
o = ambient condition
p = particular solution
p_∞ = particular solution at infinity

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